Macroeconomics Qualifying Exam – 303 Module
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Answer **two** of the following three questions. Put **each answer** in a separate blue book with the question number and your assigned exam number on it.

1. Consider the discrete-time optimal growth model with an infinitely lived representative agent. Use the following notation. Let $U(c)$ be the utility from per capita consumption c, with $\beta \in (0,1)$ the subjective discount factor. Output is produced using a neoclassical production function, $f(k)$ with k being the per capita capital stock with depreciates at rate $\delta \in [0,1]$. Assume that population grows geometrically, $N_{t+1} = (1+n)N_t$, for $n > -1$ and $N_0 = 1$.

a) Set up the planning problem and solve for the Euler equation. State the complete set of optimality conditions.

b) Find all steady state equilibria. Draw a phase portrait and derive arrows of motion.

c) Demonstrate rigorously that for some value of k_0, there is a unique growth path leading to the interior steady state. If this condition is *not* met, what is the long-run behavior of this economy? (Hint: demonstrate your answer using the results of part (b).)

d) What is the golden rule allocation? Find it for this model. Prove or disprove: The capital stock at the steady state of the planning problem is less than the golden rule level of capital stock.

2. Consider an overlapping generations model with productive capital where agents live two periods and solve

$$Max_{c_{0,t}, c_{1,t+1}} U(c_{0,t}, c_{1,t+1})$$

s.t.

$$\begin{aligned} c_{0,t} &= w_t - s_t \\ c_{1,t+1} &= R_{t+1} s_t \end{aligned}$$

where c_i is consumption at age $i = 0, 1$, t is time, w is wage, s is savings, $R = 1 + r - \delta$ is the net return on savings, where $\delta \in [0,1]$ is the depreciation rate on capital. Assume that is population is constant and normalized to one.

a) Derive consumer and firm optimization conditions. What is/are the state variable(s)? What is/are the choice variable(s)?

b) Carefully and completely define a competitive equilibrium for this model. State a theorem that guarantees that there is a unique interior steady state for this model. In one or two sentences, explain the behavioral meaning of this theorem.

c) How many stationary equilibria are there in this model? Identify each one. Draw the phase portrait assuming the theorem in part (b) holds, and include arrows of motion showing the dynamics of the system. State the stability properties of each steady state.

d) Redraw the phase portrait of this economy when utility is Cobb-Douglas, $U(c_0, c_1) = c_0^{1-\beta} c_1^{\beta}$, and wages are given by $w(k) = Ak$, for $A > 0$, and $\beta \in (0,1)$. Characterize the dynamics of this economy by drawing a new phase portrait and *deriving* arrows of motion. (Hint: there are two cases to consider, so you should draw two phase portraits). Discuss the implications of this version of the model, especially whether its implications are empirically relevant.

3. Consider a two-period OLG model with taxes. Let utility be logarithmic, $U(c_0, c_1) = \ln c_0 + \beta \ln c_1$, for $\beta > 0$, and production be Cobb-Douglas, $F(K, N) = AK^{\alpha}N^{1-\alpha}$, where K is capital, N is the working population, and the constant $A > 0$. Leisure is not valued, and the old are retired. Population is constant and you may normalize it to 1.

a) Set up a model in which there are two types of taxes which are paid only in the period of life when agents work. Let the lump-sum tax be denoted τ, and the rate of tax on first period consumption be σ. All tax revenue is unproductive. Solve for consumer and firm optima and construct the capital market equilibrium condition.

b) Suppose τ is positive but small. How many steady states are there in this model? Draw a phase portrait illustrating your answer. Prove or disprove: There exists a single value of $\tau > 0$ such that there is a unique interior steady state for this model.

c) Draw a phase portrait showing the effect on the dynamics of an increase in the consumption tax, σ. Does a change in σ affect the number of steady state equilibria? Prove or disprove: *If $\beta = \sigma - 1$, then no equilibrium exists.*

d) Consider a variant of the model in which there is no consumption or lump-sum taxes ($\sigma = \tau = 0$), and instead labor income is taxed at rate, $\eta > 0$. Further, the income tax is used to fund public investment λ, where output is now given by $K^{\alpha}\lambda^{1-\alpha}$. Derive the capital market clearing condition. Next, write down the government budget constraint at time t in terms of capital, and solve for public investment, λ, in terms of K. Substitute out λ in terms of K in the market clearing condition and draw a phase portrait for this model, including arrows of motion and identifying all steady states. (Hint: there are two cases to consider, so you should draw two phase portraits).