Macroeconomics Qualifying Exam–Sketch of Solutions
Claremont Graduate School
Spring, 1996

1. a) Agents solve

$$Max_c \Sigma_{t=0}^{\infty} \beta^t U(c_t)$$

s.t.

$$c_t = e_t + (p_t + d_t)\gamma_{t-1} - p_t\gamma_t$$

with $e_0 > 0$, $\gamma_0 = 0$ given. The asset pricing equation is found from the FOC

$$p_t U'(c_t) = \beta U'(c_{t+1})[p_{t+1} + d_{t+1}].$$

b) The agent's problem in an endowment/loan economy is

$$Max_c \Sigma_{t=0}^{\infty} \beta^t U(c_t)$$

s.t.

$$c_t = e_t + R_t s_{t-1} - s_t$$

with $s_0 = 0$ given. The FOC is

$$U'(c_t) = \beta U'(c_{t+1}) R_{t+1}$$

which is identical to that in (a) when R is defined properly. Equilibrium in the model with a stock market is a set of prices $\{p_t\}_{t=0}^{\infty}$ such that agents optimize using the FOC in (a) given $\{e_t\}_{t=0}^{\infty}$ and for each time t, $\gamma_t = 0$. Equilibrium in the representative agent loans model is acheived when agents optimize using the FOC in (b) given $\{e_t\}_{t=0}^{\infty}$ and prices $\{R_t\}_{t=1}^{\infty}$ clear the loan market so that $s_t = 0\ \forall t$.

c) Perfect consumption smoothing implies $R_{t+1} = \frac{1}{\beta}\ \forall t$. The stochastic problem is

$$Max_c E \Sigma_{t=0}^{\infty} \beta^t U(c_t)$$

s.t.

$$c_t = e_t + (p_t + d_t)\gamma_{t-1} - p_t\gamma_t.$$

The FOC is

$$p_t U'(c_t) = \beta E U'(c_{t+1})[p_{t+1} + d_{t+1}].$$

Define $x_{t+1} = \frac{U'(c_{t+1})}{U'(c_t)}$ and assuming R and x are iid random variables, the FOC above can be written as

$$E(R_{t+1}) = \frac{1}{\beta E(x_{t+1})} - \frac{Cov(x_{t+1}, R_{t+1})}{E(x_{t+1})}.$$

Using this formula, a desirable stock (high p_t) is one whose return in inversely related to the growth in consumption x. All agents realize this which drives down this stock's return.

d) The consumption CAPM yields a similar asset pricing equation,

$$E(r_{t+1}^i) = r_{t+1}^f + \beta^i E(r_{t+1}^m - r_{t+1}^f),$$

where r^i, r^f, r^m are the rate of return on stock i, the risk-free rate and the market rate, respectively with the last of these assumed perfectly negatively correlated with $U'(c_{t+1})$ and $\beta^i \equiv \frac{Cov(r^m_{t+1}, r^i_{t+1})}{Var(r^m_{t+1})}$.

2 a) Solve

$$Max_{c_0^h, c_1^h} U(c_{0,t}^h, c_{1,t+1}^h)$$

s.t.

$$\begin{aligned} c_{0,t}^h &= w_t^h - s_t^h \\ c_{1,t+1}^h &= R_{t+1} s_t^h, \end{aligned}$$

where c_0^h, c_1^h are first and second period consumption of a type h household, $R_{t+1} = 1 + r_{t+1} - \delta$ with the second and third terms denoting the MPK and the depreciation rate of capital and s is savings. Optimality is given by

$$U_1(c_{0,t}^h, c_{1,t+1}^h) = U_2(c_{0,t}^h, c_{1,t+1}^h) R_{t+1}$$

b) A competitive equilibrium is a set of prices $\{w_t^h, R_{t+1}\}_{t=0}^{\infty}$ which are profit maximizing for firms for $h \in \{1, 2, ..., H\}$; given prices consumers optimize using the FOC above; and for each time t the price R_{t+1} clears the capital market,

$$\Sigma_{h=1}^{H} N_t^h s_t^h = K_{t+1}$$

where I've redefined N^h to be the number of *young* consumers of type h. A nontrivial competitive equilibrium exists if the production and utility functions satisify the Inada conditions $\forall h, t$ and initial capital $K_0 > 0$.

c) Discounting future utility by $\frac{\beta}{1-\beta}$ (any discount factor can be used, not only this one). $U(c_0^h, c_1^h) = (1-\beta)\ln c_0^h + \beta \ln c_1^h$ and the FOC implies that

$$s_t^h = \beta w_t^h.$$

Note that the aggregate production function can be written as $F(K, N) = K^\alpha N^{1-\alpha}$ so that firm optimization leads to

$$w^h = (1-\alpha) K^\alpha N^{1-\alpha}$$

$\forall h$, so that aggregate savings ($s \equiv \Sigma_{h=1}^{H} s^h N^h$) is

$$s_t = \beta(1-\alpha) K^\alpha N^{2-\alpha}.$$

The equilibrium condition is

$$K_{t+1} = \beta(1-\alpha) K^\alpha N^{2-\alpha}.$$

The phase portrait in $K_t - K_{t+1}$ space is a standard-looking diagram, beginning at the origin, increasing and strictly concave.

d) Household type 1's problem is

$$Max_{c_0^1, c_1^1} U(c_{0,t}^1, c_{1,t+1}^1)$$

s.t.

$$\begin{aligned} c_{0,t}^1 &= w_t^1 - s_t^1 - \tau \\ c_{1,t+1}^1 &= R_{t+1} s_t^1, \end{aligned}$$

which delivers the optimal savings function $s^1 = \beta(w^1 - \tau)$. For household type 2 the problem solved is

$$Max_{c_0^2, c_1^2} U(c_{0,t}^2, c_{1,t+1}^2)$$

s.t.

$$\begin{aligned} c_{0,t}^2 &= w_t^2 - s_t^2 - \sigma \\ c_{1,t+1}^2 &= R_{t+1} s_t^2, \end{aligned}$$

with the FOC leading to $s^2 = \beta(w^2 + \sigma)$. Note that $\sigma = \frac{N^1 \tau}{N^2}$ and the equilibrium condition is $K_{t+1} = N^1 s^1 + N^2 s^2$. Using these two facts, it is easy to show that such a tax and transfer scheme has no effect on aggregate output, savings or interest rates relative to the no-tax-transfer case. Why? Identical preferences for consumption dominate the different numbers of agents in each group.

3 a) Consumer optimum is $s_t = \beta w_t$; Firm optimization implies that $w_t = (1-\alpha)k_t^\alpha$ and $r_t = \alpha k_t^{\alpha-1}$ where $k_t \equiv \frac{K_t}{N_t}$. Asset market clearing in per capita terms is

$$(1+n)(k_{t+1} + b_{t+1}) = \beta(1-\alpha)k_t^\alpha$$

where $b_t \equiv \frac{B_t}{N_t}$. The state variables are b_t, k_t at time t.

b) Steady states are $(0,0)$, $(\bar{k}, \bar{b})$, $(\bar{\bar{k}}, 0)$, where $\bar{k} = [\frac{\alpha}{\delta+n}]^{\frac{1}{1-\alpha}}$, $\bar{b} = \frac{\beta(1-\alpha)}{1+n}\bar{k}^\alpha - \bar{k}$ and $\bar{\bar{k}} = \frac{\beta(1-\alpha)}{1+n}$. Solving $k_{t+1} \geq k_t$ implies the KK line is $b_t \leq \frac{\beta(1-\alpha)}{\alpha}k_t - (1+n)k_t^{2-\alpha}$ and the BB line is found by simplifying $b_{t+1} \geq b_t$ implies $k_t \leq \bar{k}$. The phase diagram is standard.

c) Using the phase portrait, it can be seen that $(0,0)$ is unstable, $(\bar{k}, \bar{b})$ is saddle-point stable (i.e. unstable but with a one-dimension subspace which has a stable manifold) and $(\bar{\bar{k}}, 0)$ is stable. The phase portrait for $b_t = 0\ \forall t$ is the horizontal axis of the standard diagram. Thus it is clear that government debt crowds out private investment and for any $k_0 > 0$ on the saddle path, the steady state levels of output and consumption are lower with government debt. Nothing can be said about growth rates.

d) Since $\bar{k} < \bar{\bar{k}}$ it must be the case that $R(\bar{k}) > R(\bar{\bar{k}})$ implying that government debt raises interest rates by absorbing private savings. If debt is very high (in the north-west portion of the phase diagram) then government debt may eventually drive private investment to zero and the economy will crash.